Algebraic approach to directed stochastic avalanches

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A two-dimensional directed stochastic sandpile model is studied analytically with the use of directed Abelian algebras recently introduced by Alcaraz and V. Rittenberg [Phys. Rev. E **78**, 041126 (2008)]. Exact expressions for the probabilities of all possible toppling events which follow the transfer of arbitrary number of particles to a site in the stationary configuration are derived. A description of the virtual-time evolution of directed avalanches on two dimensional lattices is suggested. Due to intractability of the general problem, the algebraic approach is applied only to the solution of the special cases of directed deterministic avalanches and trivial stochastic avalanches describing simple random walks of two particles. The study of these cases has clarified the role of each particular kind of toppling in the process of avalanche growth. In the general case of the quadratic directed algebra we have determined exactly the maximum possible values of: (1) the current of particles at any given moment of virtual time and (2) the occupation number ('height') of each site at any moment of time.

I. INTRODUCTION

Sandpile models, introduced in 1987 by Bak, Tang and Wiesenfeld (BTW), have drawn a lot of attention as the simplest systems which describe Self-Organized Criticality with intrinsic avalanche-like dynamics resembling the one observed in nature [1]. Despite their simplicity and the great efforts invested in their solution, a rigorous derivation of the critical exponents describing the stationary state of the isotropic BTW models is still lacking. However, the establishment of the Abelian property of the particle topplings in the critical height

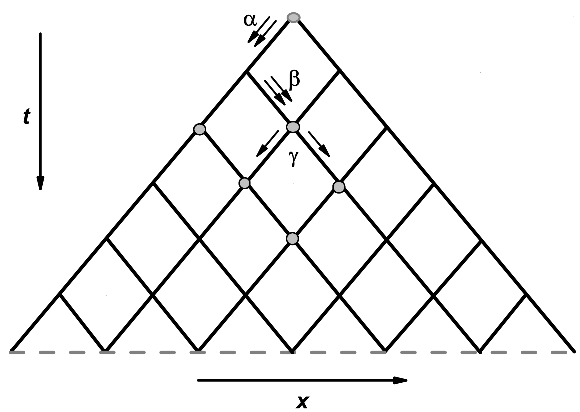

FIG. 1: Schematic representation of the rotated by $\pi/4$ square lattice and the directed toppling rules. The bottom boundary of the lattice is open.

models enhanced their analytical tractability [2]. A number of important characteristics of the stationary state have been rigorously derived [3–5]. Next, the deterministic directed sandpiles (DDS) were introduced and analytically solved [6]. It became evident that they belong to a special universality class with exactly known critical exponents.

Isotropic, as well as directed, sandpile models with stochastic dynamics were introduced too [7, 8, 12]. The numerical evaluation of the critical exponents of the stochastic directed sandpiles (SDS) has shown that they belong to a still different universality class [8, 12].

Recent extensive Monte Carlo simulations, performed by Alcaraz and Rittenberg [9] on the rotated by $\pi/4$ square lattice, see Fig. 1, have indicated that the two dimensional directed stochastic sandpiles belong to a universality class with $\sigma_\tau = 1.780 \pm 0.005$. If the estimated error bars are correct, then this result is in contradiction with the analytical prediction $\sigma_\tau = 1.75$ [10, 11], as well as with the previous numerical estimates [12]. Therefore, reexamination of the critical exponents of directed stochastic avalanches becomes important. An attempt in that direction was undertaken in [13].

A new approach to the analytical treatment of directed avalanches has been suggested by Alcaraz and Rittenberg [9]. It is based on the study of directed Abelian algebras (DAAs) on two-dimensional acyclic lattices. The quadratic algebra suggested for the lattice shown

in Fig. 1 acts in the bulk as

$$a_{i,j}^2 = \alpha[\mu a_{i+1,j}^2 + (1-\mu)a_{i,j}a_{i+1,j}] + (1-\alpha)[\mu a_{i,j+1}^2 + (1-\mu)a_{i,j}a_{i,j+1}], \tag{1}$$

where $a_{i,j}$ is the generator attached to each site (i,j) of the lattice. Here the labeling of the lattice sites is such that $i-1$ is the distance (in lattice spacings) from the right boundary, and $j-1$ is the distance from the left boundary. Thus, the nearest neighbors of site (i,j) in the direction of propagation (downwards) are $(i+1,j)$ (the left neighbor) and $(i,j+1)$ (the right neighbor).

Equation (1) describes toppling of particles from an unstable site (i,j) which involves the following stochastic events. (1) Two particles topple with probability μ: they both go to the left (right) nearest neighbor in the direction of propagation with probability $\alpha\mu$ (resp., $(1-\alpha)\mu$). (2) One particle topples and the other remains at the same site with probability $1-\mu$: the toppled particle goes to the left (right) nearest neighbor with probability $\alpha(1-\mu)$ (resp., $(1-\alpha)(1-\mu)$).

II. THE STOCHASTIC DIRECTED QUADRATIC ALGEBRA

Here we shall study a SDS on the rotated square lattice, see Fig. 1, with more simple and convenient for theoretical investigation toppling rules, a particular case of which was considered in [10] and [11]. According to these rules, any unstable site relaxes to a stable configuration (with at most one particle) through a succession of two-particle topplings: the two particles are transferred to the left (right) nearest neighbor in front with probability α (resp., β), or one goes left and the other goes right with probability $\gamma = 1-\alpha-\beta$. Each lattice site can emit only an even number of particles but can receive any number of them. Therefore, one can readily classify the sites with respect to their effect on the flux of particles [10, 11]. A site that receives an even number of particles emits the same number of them, hence, it does not change the flux and is called *passive*. A site that receives an odd number $2n+1$ of particles is *active*, since if empty it retains one particle and emits the remaining even part $2n$ of them (*negatively active*), while if occupied it emits the total number of $2n+2$ particles, i.e. increases the flux by one unit (*positively active*). It was shown that the critical state of the above SDS is a product measure with average particle density $\rho = 1/2$.

Next, we find it convenient to label the sites so that the first coordinate i is the integer

time step τ, and the second coordinate j numbers the sites which can be visited by the avalanche at time $\tau = i$ in the horizontal (spatial) direction. Thus, the lattice sites form the triangular array $\mathcal{L} = \{(i,j) : j = 1,2,\ldots,i,\ \ i = 1,2,\ldots,T\}$, where T is the size of the lattice in the temporal direction. In the above notation, the quadratic algebra we study reads

$$a_{i,j}^2 = \alpha a_{i+1,j}^2 + \beta a_{i+1,j+1}^2 + \gamma a_{i+1,j} a_{i+1,j+1}. \tag{2}$$

Here it is assumed that sites (i,j) do not belong to the open boundary of the lattice $\partial\mathcal{L} = \{i = T,\ j = 1,2,\ldots,T\}$. There are as many algebraic relations (2) as sites in the lattice. With the sites lying on the open boundary one associates generators satisfying the following T relations (see Eq. (71) in [9])

$$a_{T,j}^2 = 1, \quad j = 1,2,\ldots,T. \tag{3}$$

In the case under study the critical stationary state of the system is (see Eq. (73) in [9] at $\mu = 1$)

$$\Phi_{1,T} = \prod_{i=1}^{T} \prod_{j=1}^{i} \frac{1 + a_{i,j}}{2}. \tag{4}$$

This corresponds to a product measure with equal probability of having a site vacant or occupied by just one particle. One can show by finite induction that

$$a_{i,j}\Phi_{1,T} = \Phi_{1,T}, \quad (i,j) \in \mathcal{L}. \tag{5}$$

Avalanches will always be started by dropping particles on site $(1,1)$ until it becomes unstable. If we introduce a restriction of the stationary state to the time interval from $i = \tau$ to $i = T$,

$$\Phi_{\tau,T} = \prod_{i=\tau}^{T} \prod_{j=1}^{i} \frac{1 + a_{i,j}}{2}, \tag{6}$$

the beginning of an avalanche will be described as

$$a_{1,1}^2 \Phi_{2,T} = (\alpha a_{2,1}^2 + \beta a_{2,2}^2 + \gamma a_{2,1} a_{2,2}) \frac{1 + a_{2,1}}{2} \frac{1 + a_{2,2}}{2} \Phi_{3,T}. \tag{7}$$

One can read from here the obvious fact that the avalanche may stop at the second time-step $\tau = 2$ with probability $\gamma/4$: if the two particles on the initial site go to different neighbors (with probability γ) and both of these neighbors are empty (with probability $1/4$).

In order to compute the probabilities of larger avalanches, one has to determine the effect of an arbitrary number of particles piled up on a given site at time τ on the restriction of the

stationary state to the interval $[\tau, T]$. In particular, one has to compute for any integer n the product $a_{i,j}^n(1+a_{i,j})/2$. Since the result depends on the parity of n, we consider separately $n = 2p$ even,

$$a_{i,j}^{2p}\frac{1+a_{i,j}}{2} = \frac{1}{2}\sum_{k=0}^{2p} C_k^{(2p)}(1+a_{i,j})a_{i+1,j}^{2p-k}a_{i+1,j+1}^k, \tag{8}$$

and $n = 2p+1$ odd,

$$a_{i,j}^{2p+1}\frac{1+a_{i,j}}{2} = \frac{1}{2}\sum_{k=0}^{2p} C_k^{(2p)}a_{i,j}a_{i+1,j}^{2p-k}a_{i+1,j+1}^k + \frac{1}{2}\sum_{k=0}^{2p+2} C_k^{(2p+2)}a_{i+1,j}^{2p+2-k}a_{i+1,j+1}^k. \tag{9}$$

In the former case, when an even number $2p$ of particles comes to a stationary site (i, j), all the $2p$ particles topple: $2p - k$ to the left and k to the right with probability $C_k^{(2p)}$, $k = 0, 1, \ldots, 2p$. At that the state of the site (i, j) remains unchanged.

In the latter case, when an odd number $2p+1$ of particles comes to a stationary site (i, j), the result depends on the occupation number of that site. If the site is empty, only $2p$ particles topple: $2p-k$ to the left and k to the right with probability $C_k^{(2p)}$, $k = 0, 1, \ldots, 2p$, and one particle remains on that site. If the site is occupied, all the $2p+2$ particles will topple: $2p+2-k$ to the left and k to the right with probability $C_k^{(2p+2)}$, $k = 0, 1, \ldots, 2p+2$, and the site remains empty. However, the net state of the site (i, j) will not change, because the stationary probabilities of being empty or occupied by one particle are equal.

By deriving recurrent relations for the coefficients $C_k^{(2p)}$,

$$C_k^{(2p+2)} = \alpha C_k^{(2p)} + \beta C_{k-2}^{(2p)} + \gamma C_{k-1}^{(2p)}, \quad k = 2, \ldots, p, \tag{10}$$

and solving them under the initial conditions: $C_0^{(0)} = 1$, $C_0^{(2)} = \alpha$, $C_1^{(2)} = \gamma$, and $C_2^{(2)} = \beta$, we obtain

$$C_k^{(2p)}(\alpha,\beta,\gamma) = \sum_{m=0}^{[k/2]}\frac{p!}{(k-2m)!m!(p+m-k)!}\alpha^{p+m-k}\beta^m\gamma^{k-2m}, \quad k = 0, 1, \ldots, p, \tag{11}$$

where $[m]$ denotes the entire part of the real number m. Together with the left-right symmetry property

$$C_k^{2p}(\alpha,\beta,\gamma) = C_{2p-k}^{2p}(\beta,\alpha,\gamma),$$

Eq. (11) completely defines the coefficients $C_k^{2p}(\alpha,\beta,\gamma)$, $k = 0, 1, \ldots, 2p$. Here are their explicit expressions for $p = 2, 3$:

$$C_0^{(4)} = \alpha^2, \quad C_1^{(4)} = 2\alpha\gamma, \quad C_2^{(4)} = 2\alpha\beta+\gamma^2, \quad C_3^{(4)} = 2\beta\gamma, \quad C_4^{(4)} = \beta^2$$

$$\begin{aligned} C_0^{(6)} &= \alpha^3, \quad C_1^{(6)} = 3\alpha^2\gamma, \quad C_2^{(6)} = 3\alpha^2\beta + 3\alpha\gamma^2, \quad C_3^{(6)} = 6\alpha\beta\gamma + \gamma^3, \\ & C_4^{(6)} = 3\alpha\beta^2 + 3\beta\gamma^2, \quad C_5^{(6)} = 3\beta^2\gamma, \quad C_6^{(6)} = \beta^3. \end{aligned} \tag{12}$$

Note that $C_k^{(2p+1)} = 0$, $k = 0, 1, \ldots, 2p+1$.

In order to describe the size distribution of the avalanche, we need to know only the number of particles transferred from time-step τ to time-step $\tau + 1$. The probability with which that number vanishes for the first time is the probability of having avalanches with time duration τ. Another distribution we are interested in is the probability distribution of having avalanches with a given total number of toppled particles, which measures the "size" of an avalanche. In all these cases we are not interested in the configuration changed by the avalanche. Following Alcaraz and Rittenberg [9], we use the symbol $\hat{=}$ to denote expressions in which all the generators of the algebra left behind the front of the avalanche are replaced by unity. For example,

$$a_{i,j}^{2p}\frac{1 + a_{i,j}}{2} \hat{=} \sum_{k=0}^{2p} C_k^{(2p)} a_{i+1,j}^{2p-k} a_{i+1,j+1}^{k}, \tag{13}$$

$$a_{i,j}^{2p+1}\frac{1 + a_{i,j}}{2} \hat{=} \frac{1}{2}\sum_{k=0}^{2p} C_k^{(2p)} a_{i+1,j}^{2p-k} a_{i+1,j+1}^{k} + \frac{1}{2}\sum_{k=0}^{2p+2} C_k^{(2p+2)} a_{i+1,j}^{2p+2-k} a_{i+1,j+1}^{k}. \tag{14}$$

Let $P(n_1, \ldots, n_\tau|\tau)$ denote the probability that at time $i = \tau$ the sites $\{(\tau, 1), (\tau, 2), \ldots, (\tau, \tau)\}$ have occupation numbers $\{n_1, n_2, \ldots, n_\tau\}$, respectively. The total number of particles at that moment may range from 0 to some finite $n_{\max}(\tau)$. Now, the virtual-time evolution of avalanches on the lattice $\mathcal{L}$ is described by

$$a_{1,1}^2 \Phi_{2,T} \hat{=} \sum_{n=0}^{n_{\max}(\tau)} \left[\sum_{n_1+\cdots+n_\tau = n} P(n_1, \ldots, n_\tau|\tau) \prod_{k=1}^{\tau} a_{\tau,k}^{n_k} \right] \Phi_{\tau+1,T}\ , \quad \tau = 2, \ldots, T-1. \tag{15}$$

Of course, the avalanche continues from moment τ to moment $\tau + 1$ only if there is at least one $n_k \geq 2$. In this case the monomial $\prod_{k=1}^{\tau} a_{\tau,k}^{n_k}$ represents a possible distribution of the particles in the row τ, and the flux of particles that hits the next row $\tau + 1$ is obtained by applying formulas (13) or (14) to each $a_{\tau,k}^{n_k}$ with $n_k \geq 2$. The span of the avalanche front at the moment τ is from $j_{\min}(\tau)$ to $j_{\max}(\tau)$, where $j_{\min}(\tau)$ ($j_{\max}(\tau)$) is the leftmost (rightmost) unstable site.

The analysis of the avalanche evolution from Eq. (15) for large times τ seems untractable problem. In order to get some insight about the role of the different toppling processes, we pass to the consideration of two extreme cases of the algebra (2): $\alpha = \beta = 0, \gamma = 1$ and $\alpha = \beta = 1/2, \gamma = 0$.

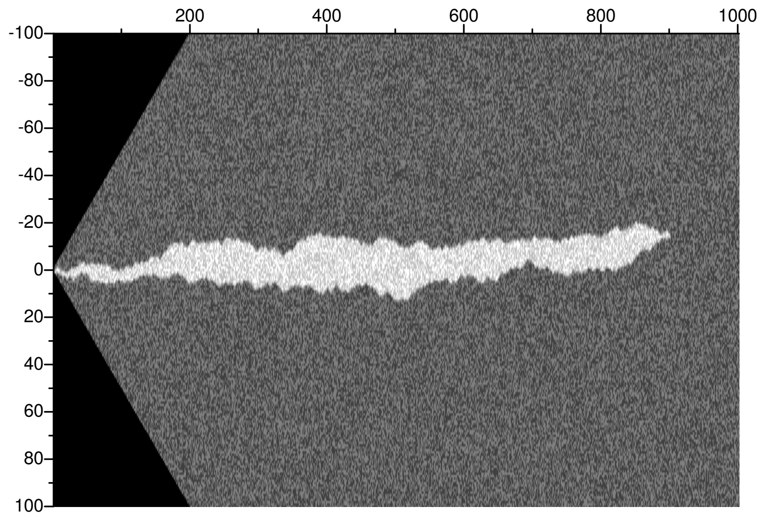

FIG. 2: Trace of a directed deterministic avalanche of duration 901 time steps.

A. Directed deterministic avalanches

When $\alpha = \beta = 0$, hence, $\gamma = 1$, the evolution of the avalanche becomes deterministic, because the two toppling particles always go to different nearest neighbors in front. Thus, the left and right boundaries of the unstable avalanche cluster perform simple random walks, see Fig. 2. In this case the coefficients (11) reduce to

$$C_k^{(2p)}(0,0,1) = \delta_{k,p}, \tag{16}$$

and relations (13), (14) simplify to

$$\begin{aligned} a_{i,j}^{2p}\frac{1+a_{i,j}}{2} &\hat{=} a_{i+1,j}^{p}a_{i+1,j+1}^{p}, \\ a_{i,j}^{2p+1}\frac{1+a_{i,j}}{2} &\hat{=} \frac{1}{2}a_{i+1,j}^{p}a_{i+1,j+1}^{p} + \frac{1}{2}a_{i+1,j}^{p+1}a_{i+1,j+1}^{p+1}. \end{aligned} \tag{17}$$

Now we shall prove that the virtual-time evolution of the deterministic avalanche, described by (15) with the toppling rules (17), simplifies drastically. The avalanche front becomes “compact” (without gaps of stable sites) and almost “flat”: the unstable sites may have only two or three particles. In the first step of the evolution this is trivially true, since

$$a_{1,1}^2\frac{1+a_{2,1}}{2}\frac{1+a_{2,2}}{2} = \frac{1}{4}(a_{2,1}a_{2,2}+a_{2,1}^2a_{2,2}+a_{2,1}a_{2,2}^2+a_{2,1}^2a_{2,2}^2)\hat{=}\frac{1}{4}(1+a_{2,1}^2+a_{2,2}^2+a_{2,1}^2a_{2,2}^2). \tag{18}$$

It is seen that with probability 1/4 the avalanche stops and, whenever it continues, the

unstable sites at $\tau = 2$ are occupied by exactly two particles. In the next step we obtain

$$a_{1,1}^2 \prod_{i=2}^{3} \prod_{j=1}^{i} \frac{1 + a_{i,j}}{2} \hat{=} \frac{1}{4}(1 + a_{3,1}a_{3,2} + a_{3,2}a_{3,3} + a_{3,1}a_{3,2}^2 a_{3,3}) \prod_{j=1}^{3} \frac{1 + a_{3,j}}{2}$$
$$\hat{=} \frac{1}{32}(12 + 2a_{3,1}^2 + 5a_{3,2}^2 + 2a_{3,3}^2 + a_{3,2}^3 + 3a_{3,1}^2 a_{3,2}^2 + 3a_{3,2}^2 a_{3,3}^2 + a_{3,1}^2 a_{3,2}^3$$
$$+ a_{3,2}^3 a_{3,2}^2 + a_{3,1}^2 a_{3,2}^2 a_{3,3}^2 + a_{3,1}^2 a_{3,2}^3 a_{3,3}^2). \tag{19}$$

Obviously, the set of unstable sites remains compact and the occupation numbers of the unstable sites equal only 2 or 3. Having established these properties for $\tau = 3$, we prove now that they persist for $\tau + 1$.

According to our assumption, at some τ the general term of the operator products in the right-hand side of (15) has the form

$$a_{\tau,1}^{n_1} \cdots a_{\tau,p}^{n_p} a_{\tau,p+1}^{n_{p+1}} \cdots a_{\tau,p+q}^{n_{p+q}} a_{\tau,p+q+1}^{n_{p+q+1}} \cdots a_{\tau,\tau}^{n_\tau}, \tag{20}$$

where $n_1, \ldots, n_p$ and $n_{p+q+1}, \ldots, n_\tau$ take values $0, 1$ and $n_{p+1}, \ldots, n_{p+q}$ equal 2 or 3. This term describes an avalanche with front spanning q adjacent unstable sites. Since each unstable site emits only two particles, after the substitution

$$a_{\tau,i} \hat{=} 1, \quad a_{\tau,i}^2 = a_{\tau+1,i}a_{\tau+1,i+1}, \quad a_{\tau,i}^3 \hat{=} a_{\tau+1,i}a_{\tau+1,i+1}, \qquad i \in \{1, 2, \ldots, \tau\}, \tag{21}$$

the terms that are relevant for generation of the avalanche evolution at time step $\tau + 1$ take the form

$$\prod_{k=1}^{\tau} a_{\tau,k}^{n_k} \Phi_{\tau+1,T} \hat{=} \prod_{k=p+1}^{p+q} a_{\tau+1,k}a_{\tau+1,k+1} \left[\prod_{i=1}^{\tau+1} \frac{1 + a_{\tau+1,i}}{2} \right] \Phi_{\tau+2,T}$$
$$= \left[\prod_{i=1}^{p} \frac{1 + a_{\tau+1,i}}{2} \right] \frac{a_{\tau+1,p+1} + a_{\tau+1,p+1}^2}{2} \left[\prod_{k=p+2}^{p+q} \frac{a_{\tau+1,k}^2 + a_{\tau+1,k}^3}{2} \right]$$
$$\times \frac{a_{\tau+1,p+q+1} + a_{\tau+1,p+q+1}^2}{2} \Phi_{\tau+2,T}$$
$$\hat{=} \left(\frac{1}{2} + \frac{1}{2} a_{\tau+1,p+1}^2 \right) \left[\prod_{k=p+2}^{p+q} a_{\tau+1,k}^2 \right] \left(\frac{1}{2} + \frac{1}{2} a_{\tau+1,p+q+1}^2 \right) \Phi_{\tau+2,T}. \tag{22}$$

From the above expressions one can tell that: (1) the avalanche front remains compact; (2) the unstable sites can have height 2 or 3 only; (3) if the hitting avalanche front is of length q at time τ, in the next moment of time $\tau + 1$ it can: (a) shrink to $q - 1$ with probability 1/4; (b) remain of the same length q with probability 1/2; (c) expand to length $q + 1$ with

probability 1/4. The latter property reflects the fact that the left and right boundary of the compact avalanche front undergo independently simple random walks one step to the left and one step to the right. At that, a step to the left neighbor in front does not change the distance from the left boundary, while a step to the right increases it by one unit. Denoting by $P(L,\tau)$ the probability of an avalanche to have front of length at time τ, one obtains the recurrence relation

$$P(L,\tau+1) = \frac{1}{4}P(L-1,\tau) + \frac{1}{2}P(L,\tau) + \frac{1}{4}P(L+1,\tau). \tag{23}$$

B. Simple random walks

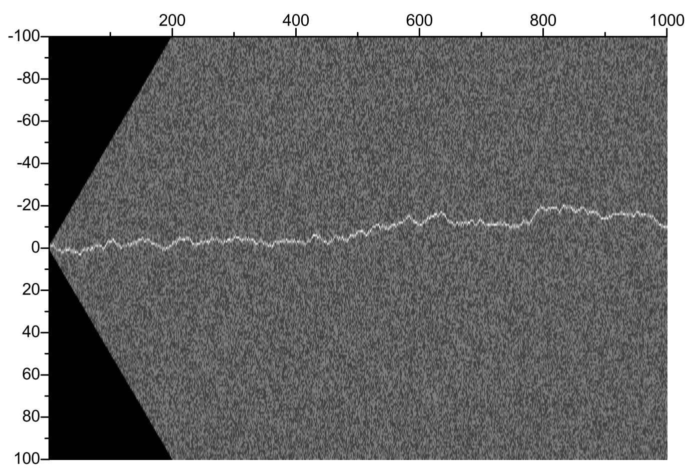

FIG. 3: Trace of a directed stochastic avalanche of the simple random walk type.

In the case $\alpha = \beta = 1/2, \gamma = 0$ we obtain avalanches carrying exactly two particles throughout the whole lattice, the trace of which represents an unbiased simple random walk, see Fig. 3. The coefficients (11) become

$$C_{2m}^{(2p)}(1/2,1/2,0) = 2^{-p}\binom{p}{m}, \quad C_{2m+1}^{(2p)}(1/2,1/2,0) = 0, \quad m = 0,1,\ldots,p-1, \tag{24}$$

and relations (13), (14) take the form

$$a_{i,j}^{2p}\frac{1+a_{i,j}}{2} \hat{=} 2^{-p}\sum_{m=0}^{p}\binom{p}{m} a_{i+1,j}^{2(p-m)} a_{i+1,j+1}^{2m},$$

$$a_{i,j}^{2p+1}\frac{1+a_{i,j}}{2} \hat{=} 2^{-p-1}\sum_{m=0}^{p}\binom{p}{m}a_{i+1,j}^{2(p-m)}a_{i+1,j+1}^{2m} + 2^{-p-2}\sum_{m=0}^{p+1}\binom{p+1}{m}a_{i+1,j}^{2(p+1-m)}a_{i+1,j+1}^{2m}. \quad (25)$$

Now the quadratic algebra (2) has two terms

$$a_{i,j}^2 = \frac{1}{2}a_{i+1,j}^2 + \frac{1}{2}a_{i+1,j+1}^2 \quad (26)$$

which act on the stationary state independently, without changing it. We shall prove that the front of each avalanche now has just one unstable site occupied by 2 or 3 particles. The avalanches perform a simple random walk and end up at the open boundary of the system at $\tau = T$.

In the initial step of the avalanche evolution, one has

$$\begin{aligned} &a_{1,1}^2\frac{1+a_{2,1}}{2}\frac{1+a_{2,2}}{2} = \frac{1}{8}(a_{2,1}^2 + a_{2,1}^3 + a_{2,1}^2a_{2,2} + a_{2,1}^3a_{2,2} + a_{2,2}^2 + a_{2,1}a_{2,2}^2 + a_{2,2}^3 + a_{2,1}a_{2,2}^3) \\ &\hat{=} \frac{1}{2}a_{2,1}^2 + \frac{1}{2}a_{2,2}^2 = \frac{1}{4}(a_{3,1}^2 + 2a_{3,2}^2 + a_{3,3}^2). \quad (27) \end{aligned}$$

It is seen that with equal probability 1/2 the two particles of the avalanche go to the left or to the right nearest-neighbor ahead. In the next step, the row $\tau = 2$ emits again exactly 2 particles, which are distributed according to the unbiased simple random walk probabilities. After hitting the stationary distribution of the row $\tau = 3$, the avalanche continues in the same way:

$$\begin{aligned} &a_{1,1}^2\prod_{i=2}^{3}\prod_{j=1}^{i}\frac{1+a_{i,j}}{2} \hat{=} \frac{1}{4}(a_{3,1}^2 + 2a_{3,2}^2 + a_{3,3}^2)\prod_{j=1}^{3}\frac{1+a_{3,j}}{2} \\ &\hat{=} \frac{1}{4}(a_{3,1}^2 + 2a_{3,2}^2 + a_{3,3}^2) = \frac{1}{8}(a_{4,1}^2 + 3a_{4,2}^2 + 3a_{4,3}^2 + a_{4,4}^2). \quad (28) \end{aligned}$$

Thus, up to $\tau = 3$, the front of each avalanche contains exactly one unstable site with occupation number equal to 2 or 3 only. The emitted particles from layer to layer are exactly 2. Having established these properties for $\tau = 3$, we prove now that they persist for $\tau + 1$.

According to our assumption, at some τ the general term of the operator products in the right-hand side of (15) has the form

$$\begin{aligned} a_{\tau,j}^2\Phi_{\tau+1,T} &= \frac{1}{2}(a_{\tau+1,j}^2 + a_{\tau+1,j+1}^2)\left[\prod_{i=1}^{\tau+1}\frac{1+a_{\tau+1,i}}{2}\right]\Phi_{\tau+2,T} \\ &= \left[\prod_{i=1}^{j-1}\frac{1+a_{\tau+1,i}}{2}\right]\frac{a_{\tau+1,j}^2 + a_{\tau+1,j}^3}{4}\left[\prod_{i=j+1}^{\tau+1}\frac{1+a_{\tau+1,i}}{2}\right]\Phi_{\tau+2,T} \end{aligned}$$

$$+\left[\prod_{i=1}^{j}\frac{1+a_{\tau+1,i}}{2}\right]\frac{a^2_{\tau+1,j+1}+a^3_{\tau+1,j+1}}{4}\left[\prod_{i=j+2}^{\tau+1}\frac{1+a_{\tau+1,i}}{2}\right]\Phi_{\tau+2,T}$$
$$\hat{=}\frac{1}{4}(a^2_{\tau+2,j}+2a^2_{\tau+2,j+1}+a^2_{\tau+2,j+2})\Phi_{\tau+2,T}. \tag{29}$$

For general values of α and $\beta = 1-\alpha$, the algebraic form of the evolution at any virtual time corresponds to processes where the two particles from site $(1,1)$ pass through the whole lattice along trajectories with probability distribution corresponding to the biased simple random walk.

C. Exact results for the stochastic avalanche

In this section we present exact results concerning some important extremal characteristics of the directed stochastic avalanches which obey the general algebra (2). We derive the maximum value $I_{\max}(\tau)$ of the current of particles and the maximum height $h_{\max}(\tau, j)$ at any site (τ, j), $j = 1, \ldots, \tau$, for any given moment of virtual time τ.

1. *The maximum current at virtual time τ*

At $\tau = 1$ the value of $I_{\max}(1) = 2$ is the initial condition for an avalanche to start. At $\tau = 2$ the value of $I_{\max}(2) = 4$ is attained when both sites $(2,1)$ and $(2,2)$ are positively active, i.e., they are occupied and receive one particle each in the toppling $a^2_{1,1} \to \gamma a_{2,1}a_{2,2}$. In general, the maximum possible current of particles $I_{\max}(\tau)$ at virtual time τ is attained whenever the row $\tau-1$ has emitted the maximum possible current $I_{\max}(\tau-1)$, and the maximum possible number of sites in the row τ are positively active, i.e., occupied and visited by an odd number of particles. Since, due to the algebra (2), the current is always even, the latter quantity obviously depends on the parity of τ. Examples of such topplings between virtual times τ = 1,2,3, and 4 are shown in Fig. 4.

First we prove the following proposition.

Proposition I. The maximum current of particles that leave the row τ is given by

$$I_{max}(\tau) = \frac{\tau^2+1}{2}+1, \qquad \text{for} \quad \tau \text{ odd}; \tag{30}$$
$$I_{max}(\tau) = \frac{\tau^2}{2}+2, \qquad \text{for} \quad \tau \text{ even}. \tag{31}$$

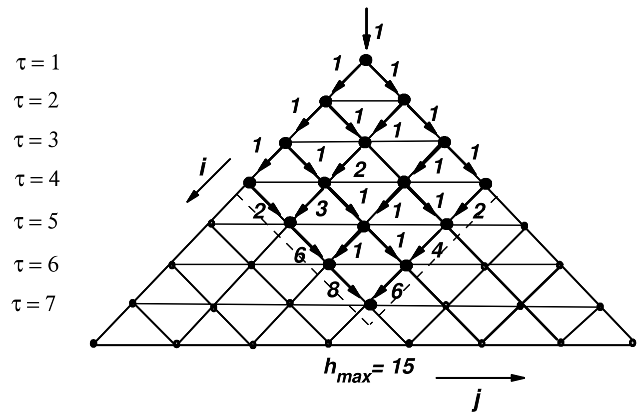

FIG. 4: Schematic illustration of an avalanche leading to a maximum unstable height at the central site of an odd-τ row. The integers besides the arrows indicate the number of particles transferred in the corresponding direction

Proof. The proof contains two important ingredients. The first one is to find avalanches which transfer the maximum possible number of particles from row to row. To ensure most favorable conditions, we consider stable configurations in which all the sites in the rows $1 \leq t \leq \tau$ are occupied. The second one is the derivation of recurrence relation between $I_{\max}(\tau)$ and $I_{\max}(\tau-2)$. The solutions of this recurrence, separately for odd and even τ, yield the desired proof.

We begin with proving that any even-τ row can have all of its sites positively active, therefore it can emit $I_{\max}(\tau-1)+\tau$ particles. Such an event occurs as a result of the following sequence of topplings. First, each of the $\tau/2$ odd-numbered sites in the row $\tau-1$, that is $\{(\tau-1, 2m-1), m=1,\ldots,\tau/2\}$, emits two particles in a process

$$a^2_{\tau-1,2m-1} \to \gamma a_{\tau,2m-1} a_{\tau,2m}, \quad m=1,\ldots,\tau/2, \tag{32}$$

so that each site of the row τ receives exactly one particle. This is possible because, for $\tau \geq 3$, the unstable configuration of the layer $\tau-1$ contains maximum $I_{\max}(\tau-2)+\tau-1$ particles: $I_{\max}(\tau-2)$ received from the preceding layer $\tau-2$, and $\tau-1$ particles at the sites in the stationary configuration of the layer $\tau-1$. After the completion of the topplings (32), in the row $\tau-1$ there will remain $I_{\max}(\tau-2)-1$ particles. The even part of that number, namely $I_{\max}(\tau-2)-2$ particles, can be transferred to row τ in arbitrary portions of even

numbers and one particle will remain in the row $\tau - 1$. Thus, the total number of particles accumulated in the unstable configuration of row τ is $I_{\max}(\tau-2)+2\tau-2$ and all of them are partitioned in even portions on it sites. Therefore,

$$I_{\max}(\tau) = I_{\max}(\tau-1)+\tau = I_{\max}(\tau-2)+2\tau-2, \quad \tau \geq 4 \text{ even}. \tag{33}$$

Next we prove that an odd-τ row can have at most $\tau-1$ positively active sites. Obviously, not all τ sites can be active, because the received number of particles $I_{\max}(\tau-1)$ is always even and cannot be distributed in odd portions among the odd number τ of sites. To prove that exactly one site can remain passive, we consider the processes (32) for $m = 1, \ldots, (\tau-1)/2$ which transfers exactly one particle to each of the first $\tau-1$ sites in the row τ. The last unstable site $(\tau-1, \tau-1)$ in the previous row can emit either an even number of particles to the site (τ, τ), thus leaving it passive, or it can send odd portions of particles to each of the sites $(\tau, \tau-1)$ and (τ, τ). In the latter case the result will be $(\tau, \tau-1)$ passive and (τ, τ) positively active. The existence of just one passive site in the row τ will not change when the remaining unstable sites in the row $\tau-1$ topple even portions of particles to sites in τ. Hence, an odd row τ can emit maximum $I_{\max}(\tau) = I_{\max}(\tau-1)+\tau-1$ particles. Since $\tau-1$ is even, by the previous argument we have $I_{\max}(\tau-1) = I_{\max}(\tau-2)+\tau-1$. Therefore,

$$I_{\max}(\tau) = I_{\max}(\tau-1)+\tau-1 = I_{\max}(\tau-2)+2\tau-2, \quad \tau \geq 3 \text{ odd}. \tag{34}$$

Summarizing, independently of the parity of τ, we obtain the recurrence relation

$$I_{\max}(\tau) - I_{\max}(\tau-2) = 2\tau-2, \qquad \tau > 2. \tag{35}$$

To solve the above recurrence, we note that for odd moments of virtual time, $\tau = 2n-1$, $n = 1, 2, \ldots$, it yields

$$\sum_{k=2}^{n}[I_{\max}(2k-1) - I_{\max}(2k-3)] \equiv I_{\max}(2n-1) - I_{\max}(1) = \sum_{k=2}^{n}(4k-4) = 2n^2-2n. \tag{36}$$

By taking into account the initial condition $I_{\max}(1) = 2$ and substituting $n = (\tau+1)/2$, we prove the first part of Proposition I.

For an even $\tau = 2n$, $n = 1, 2, \ldots$, we obtain from the recurrence (35)

$$\sum_{k=2}^{n}[I_{\max}(2k) - I_{\max}(2k-2)] \equiv I_{\max}(2n) - I_{\max}(2) = \sum_{k=2}^{n}(4k-2) = (2n)^2-2. \tag{37}$$

Then, by taking into account the initial condition $I_{\max}(2) = 4$ and substituting $n = \tau/2$, we complete the proof of Proposition I.

2. The maximum height at a site at time τ

The proof of the results presented here uses the following definition.

Definition. For each site (τ, j), $2 \le \tau \le T$, $1 \le j \le \tau$, we define a *basin of attraction* of particles as the set of all the preceding sites, which can send particles to that site by means of an avalanche obeying the algebra (2).

The principle of establishing the maximum possible occupation number of a site is to consider avalanches which realize the maximum possible current from layer to layer within the basin of attraction of the given site. Since, by intuition (at least, when $\alpha = \beta$), the largest occupation numbers for algebras (2) are reached at the middle of the row, we consider first the central sites $(\tau, (\tau+1)/2)$ for τ odd, and $(\tau, \tau/2 - 1)$, $(\tau, \tau/2 + 1)$ for τ even.

Proposition II. The maximum height attained at the central site of a row τ is:

(1) At odd moments of virtual time $\tau = 2n-1$, the maximum height at sites $(2n-1, n \pm p)$, $p = 0, 1, 2, \ldots, n-1$, is

$$h_{\max}(2n-1, n \pm p) = \begin{cases} n(n-1) - p(p-1) + 3, & \text{for} \quad n-p \ \text{ even} \\ n(n-1) - p(p+1) + 3, & \text{for} \quad n-p \ \text{ odd.} \end{cases} \tag{38}$$

The global maximum of the height is reached at the central site $(2n-1, n)$ and equals

$$h_{\max}(\tau, (\tau+1)/2) = (\tau^2 - 1)/4 + 3. \tag{39}$$

(2) At even moments of virtual time $\tau = 2n$, the maximum height at each of the sites $(2n, n-p)$, $(2n, n+p+1)$, $p = 0, 1, 2, \ldots, n-1$, is

$$h_{\max}(2n, n-p) = h_{\max}(2n, n+p+1) = \begin{cases} n^2 - p^2 + 3, & \text{for} \quad n-p \ \text{ even} \\ n^2 - p^2 - 2(p-1), & \text{for} \quad n-p \ \text{ odd.} \end{cases} \tag{40}$$

The global maximum of the height is reached at each of the central site $(2n, n)$, $(2n, n+1)$ and equals

$$h_{\max}(2n, n) = h_{\max}(2n, n+1) = \begin{cases} \tau^2/4 + 3, & \text{for} \quad \tau/2 \ \text{ even} \\ \tau^2/4 + 2, & \text{for} \quad \tau/2 \ \text{ odd.} \end{cases} \tag{41}$$

Proof. We consider again the most favorable stable configuration, which is the fully occupied basin of attraction of the given site. In the proof of Proposition I, when counting the number of particles carried downstream by an avalanche, we established the following results:

(i) An avalanche passing from even row to the next odd row can transfer ahead at most *all but one* of the particles in the fully occupied stable configuration of the odd row.

(ii) There exists an avalanche which on passing from odd row to the next even row can transfer ahead *all* of the particles in the fully occupied stable configuration of the even row.

The above statements were proved in the case when all the sites of a pair of subsequent rows can participate in the avalanche. In the present consideration we need analogous results for avalanches spreading only in the basin of attraction of a given site. Thus we encounter the problem of transfer of particles between segments of subsequent rows and we prove first the following stronger results.

Lemma 1. Consider segments of two consecutive rows which obey the condition that each site in one of the segments has at least one nearest neighbor in the other one. Then:

(a) If the number of sites in the second segment is odd, avalanches can transfer downstream at most *all but one* of the particles in the fully occupied stable configuration of that segment.

(b) If the number of sites in the second segment is even, there exists an avalanche which can transfer downstream *all* the particles in the fully occupied stable configuration of that segment.

Proof. There are three possible types of configurations of the two segments which satisfy the conditions of the lemma. With respect to the change in number of sites on passing downstream from segment to segment, these can be classified as follows:

(1) With increasing number of sites,

$$\begin{aligned}&(i,j+1),(i,j+2),\dots,(i,j+m-1)\\&(i+1,j+1),(i+1,j+2),\dots,(i+1,j+m-1),(i+1,j+m);\end{aligned}\tag{42}$$

(2) With equal number of sites,

$$\begin{aligned}&(i,j+1),(i,j+2),\dots,(i,j+m-1),(i,j+m)\\&(i+1,j+1),(i+1,j+2),\dots,(i,j+m-1),(i+1,j+m);\end{aligned}\tag{43}$$

or

$$\begin{aligned}&(i,j),(i,j+1),\dots,(i,j+m-1)\\&(i+1,j+1),(i+1,j+2),\dots,(i,j+m-1),(i+1,j+m).\end{aligned}\tag{44}$$

(3) With decreasing number of sites,

$$
\begin{aligned}
&(i,j),(i,j+1),\ldots,(i,j+m-1),(i,j+m)\\
&(i+1,j+1),(i+1,j+2),\ldots,(i,j+m-1),(i+1,j+m).
\end{aligned}
\tag{45}
$$

Due to the left-right symmetry of the lattice, when both $\alpha > 0$ and $\beta > 0$, it suffices to consider only one of the realizations of case (2), say, the configuration (44).

Note that in our notation the number of sites in the second segment is always m. With regard to the parity of m, each of the above cases splits into two subcases: (a) m odd, and (b) m even.

(a) Let m be odd. No avalanche can turn all the sites of the second segment $i+1$ into positively active ones, because every avalanche transfers downstream an even number of particles which cannot be distributed in odd portions among an odd number of sites. However, there exist an avalanche which can make all but one of the occupied sites in the segment $i+1$ positively active. An example of such an avalanche includes the topplings

$$
a_{i,k}^2 \to \gamma a_{i+1,k}a_{i+1,k+1}, \tag{46}
$$

with $k = j+1, j+3, \ldots, j+m-2$. As a result, in each of the configurations (1)-(3), all the sites of the second segment, except the last one $(i+1, j+m)$, receive exactly one particle. All the remaining unstable sites in the uper segment i can emit even portions of particles to their neighbors in segment $i+1$. Thus all $m-1$ sites $(i+1,k)$, $k = j+1, j+2, \ldots, j+m-1$, become positively active, only the last site $(i+1, j+m)$ remains passive or, possibly, stable in the cases (42) and (44).

(b) Let m be even. An avalanche, which turns all the sites of the second segment into positively active ones, is constructed as follows. After the topplings (46) with $k = j+1, j+3, \ldots, j+m-1$, all the sites in the lower segment $i+1$ receive exactly one particle. Then, the remaining unstable sites in the upper segment i transfer even portions of particles to their nearest neighbors in the downstream segment $i+1$. The existence of at least one such neighbor is ensured by the conditions of the lemma.

Now we turn back to the proof of Proposition II. Since the details of the analysis depend on the parity of τ, the different cases are considered separately.

(1) Consider first the case of odd $\tau = 2n-1$, $n = 2, 3, \ldots$. Then the basin of attraction of the central site $(2n-1, n)$ is the square with vertices at sites $(1,1)$, $(n,1)$, (n,n) and

$(2n-1,n)$, see Fig. 4. Provided an avalanche has started, the total number of particles in the basin of attraction is n^2+1. However, not all of these particles can be delivered to the target site $(2n-1,n)$, because, before reaching that site, the avalanche has to pass through $N_{\mathrm{odd}}(2n-1) = n-2$ intermediate rows with odd number of sites, leaving a particle at each of them. Therefore,

$$h_{\max}(2n-1,n) = n^2+1-(n-2) = n^2-n+3 = (\tau^2-1)/4+3. \tag{47}$$

Consider next a shift by $p = 1,2,\ldots,n-1$ sites to the left or to the right of the central site $(2n-1,n)$. Such a shift changes the basin of attraction from the square $n\times n$ to a rectangle $(n+p)\times(n-p)$. Thus, the number of particles in the fully occupied stable configuration of the basin of attraction of the sites $(2n-1,n\pm p)$ decreases to n^2-p^2. Remarkably, the new rectangular basin of attraction contains $2p+1$ segments (from row $n-p$ to row $n+p$) which have equal number $n-p$ of sites. Thus, the maximum possible number of particles transferred by an avalanche to the sites $(2n-1,n\pm p)$ depends on the parity of $n-p$, as well as on the parity of n.

(1a) When $n-p$ is even, all the particles on the $2p+1$ central even-length segments can be transferred downstream by an avalanche (see Lemma 1). Excluding the initial and final sites, there remain $2(n-p-2)$ unequal-length segments, half of which contain odd number of sites. Hence, the maximum number of particles that can occupy sites $(2n-1,n\pm p)$ is

$$h_{\max}(2n-1,n\pm p) = n(n-1)-p(p-1)+3 \qquad \text{for} \quad n-p \quad \text{even}. \tag{48}$$

Hence, in the case of maximal shift $p=n-2$ (p must be odd) one obtains

$$h_{\max}(2n-1,2) = h_{\max}(2n-1,2n-2) = 4n-3 = 2\tau-1.$$

(1b) When $n-p$ is odd, all the avalanches will leave one particle on the $2p+1$ central odd-length segments (see Lemma 1). Excluding the initial and final sites, there again remain $2(n-p-2)$ unequal-length segments. However, now between the initial (final) site and the central $2p+1$ odd-length segments there are two series of segments each of which begins and ends up with an even-length segment (the first one from 2 to $n-p-1$ and the second one from $n+p+1$ to 2). Hence, the remaining odd-length segments are $n-p-3$. Therefore, the maximum number of particles that can occupy sites $(2n-1,n\pm p)$ is

$$h_{\max}(2n-1,n\pm p) = n(n-1)-p(p+1)+3 \qquad \text{for} \quad n-p \quad \text{odd}. \tag{49}$$

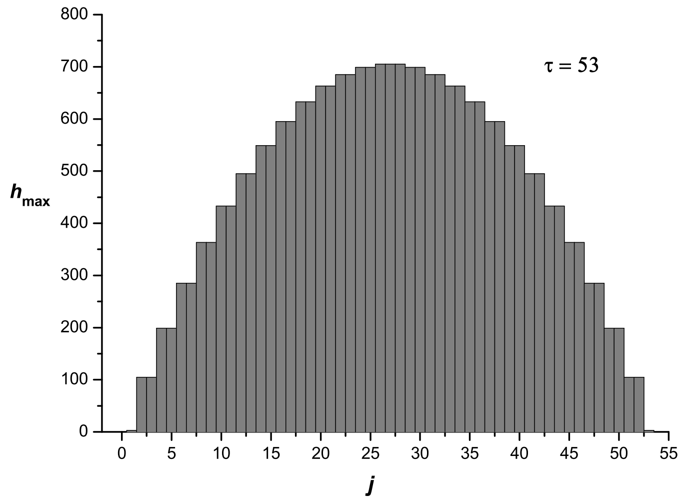

FIG. 5: Distribution of the maximum possible unstable height at a site at virtual-time moment $\tau = 53$. Since $(\tau + 1)/2$ is odd, the maximum is attained at the three central sites, although in different avalanches, see Discussion.

Note that at $p = 0$ expressions (48) and (49) yield the same result (47) for the global maximum at time $\tau = 2n - 1$, irrespectively of the parity of n. Remarkably, for n odd the substitution $p = 1$ in (48) shows that all three central sites have the same height $h_{\rm max}(2n-1, n \pm 1) = h_{\rm max}(2n-1, n)$. This case is illustrated in Fig. 5 for $n = 27$.

(2) Consider now the case of even $\tau = 2n$, $n = 2, 3, \ldots$. Due to the symmetry of the lattice, each of the two central sites $(2n, n)$ and $(2n, n+1)$ has the same maximum occupation number. For definiteness, consider the basin of attraction of the site $(2n, n+1)$. It represents a rectangle $n \times (n+1)$ with vertices at sites $(1, 1)$, $(n, 1)$, $(n+1, n+1)$ and $(2n, n+1)$, see Fig. 6. Provided an avalanche has started, the total number of particles in the basin of attraction is $n(n+1)+1$. Again, not all of these particles can be delivered to the target site $(2n, n+1)$, because, before reaching that site, the avalanche has to pass through $N_{\rm odd}(2n)$ of intermediate rows with odd number of sites, leaving a particle at each of them. The new feature here is that $N_{\rm odd}(2n)$ depends on the parity of the number n, because the intersection of row $n+1$ with the basin of attraction has the same number of sites as the row n.

(i) When n is even $N_{\rm odd}(2n) = n - 2$ and

$$h_{\rm max}(2n, n+1) = n(n+1) + 1 - (n-2) = n^2 + 3 = \tau^2/4 + 3, \quad \tau/2 \text{ even}. \qquad (50)$$

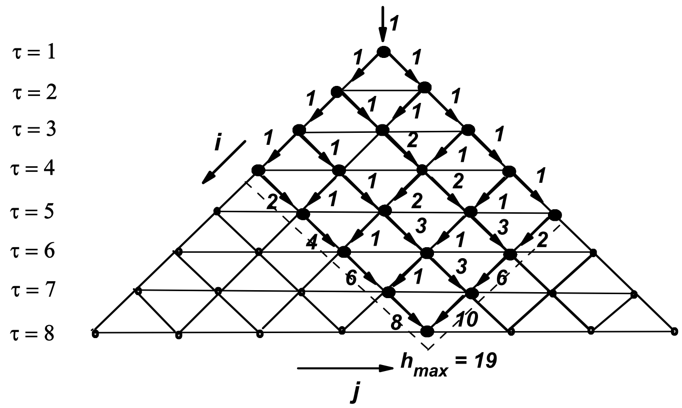

FIG. 6: Schematic illustration of an avalanche leading to a maximum unstable height at a central site of an even-τ row when $\tau/2$ is even. The integers besides the arrows indicate the number of particles transferred in the corresponding direction

This case is illustrated for $\tau = 8$ by the avalanche shown in Fig. 6

(ii) When n is odd $N_{\rm odd}(2n) = n - 1$ and

$$h_{\rm max}(2n, n+1) = n(n+1) + 1 - (n-1) = n^2 + 2 = \tau^2/4 + 2, \quad \tau/2 \text{ odd}. \tag{51}$$

This case is illustrated for $\tau = 6$ by the avalanche shown in Fig. 7.

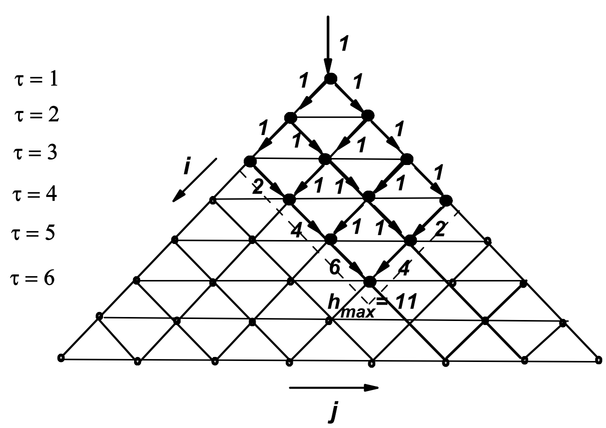

FIG. 7: The same as in Fig. 6 for even-τ row when $\tau/2$ is odd.

Consider next a shift by $p = 1, 2, \ldots, n-1$ sites to the left of the left central site $(2n, n)$ or to the right of the right central site $(2n, n+1)$. Such a shift changes the basin of attraction from the rectangle $n \times (n+1)$ to a rectangle $(n-p) \times (n+p+1)$. Thus, provided an avalanche has started, in the fully occupied stable configuration of the basins of attraction of the sites $(2n, n-p)$ and $(2n, n+p+1)$ there are $(n-p)(n+p+1)+1$ particles. Note that from row $n-p$ up to row $n+p+1$ there are $2p+2$ segments with equal number of sites $n-p$. Thus, the number of segments with odd number of sites in the basin of attraction depends on the parity of $n-p$.

(2a) When $n-p$ is even, all the particles from the $2p+2$ central even-length segments can be transferred downstream by an avalanche. Excluding the initial and final sites, there remain $2(n-p-2)$ intermediate unequal-length segments, half of which contain odd number of sites. Therefore, the maximum number of particles that can occupy sites $(2n, n-p)$ and $(2n, n+p+1)$ with $n-p$ even is

$$(n-p)(n+p+1)+1-(n-p-2) = n^2 - p^2 + 3.$$

(2b) When $n-p$ is odd, all avalanches leave one particle on the $2p+2$ central odd-length segments. Excluding the initial and final sites, there remain again $2(n-p-2)$ intermediate unequal-length segments: $n-p-2$ with length increasing from 2 up to $n-p-1$ and $n-p-2$ with length decreasing from $n-p-1$ down to 2. Since each of the above series of segments begins and ends with an even-length segment, the number of intermediate segments with odd number of sites is $n-p-3$. Therefore, the maximum number of particles that can occupy sites $(2n, n-p)$ and $(2n, n+p+1)$ with $n-p$ odd is

$$(n-p)(n+p+1)+1-(n-p-3)-(2p+2) = n^2 - p^2 - 2(p-1).$$

This completes the proof of Proposition II.

III. DISCUSSION

Here we have made an attempt to use the directed Abelian algebras, recently introduced by Alcaraz and Rittenberg [9], in the study of directed avalanches with stochastic toppling rules on the rotated square lattice. We have considered the directed quadratic algebra (2) which corresponds exactly to the stochastic toppling rules of the avalanches analytically

studied in [10] and [11]. Within different continuous approximations, the latter works predicted a consistent set of critical exponents which was questioned by the large-scale computer simulations in [9].

We have derived exact expressions for the probabilities (11) of all possible toppling events which follow the transfer of arbitrary number of particles to a site in the stationary configuration, see (8), (9). We have suggested a description (15) of the virtual-time evolution of directed avalanches on two dimensional lattices from which, in principle, the probability distribution of avalanche durations can be derived. However, the solution of the problem for large times seems untractable.

We succeeded in applying the algebraic approach only to the extreme cases of directed deterministic avalanches (when $\alpha = \beta = 0$ in (2)) and trivial stochastic avalanches describing simple random walks of two particles (when $\alpha, \beta > 0$ and $\gamma = 0$ in (2)). However, the study of these cases has clarified the role of each particular kind of toppling in the process of avalanche growth. For example, the process which ensures both the avalanche growth and decay is the toppling of an unstable site which transfers odd number of particles to each of its nearest neighbors ahead: if both of these neighbors are occupied (empty), the number of particles in the avalanche increases (decreases) by two. On the other hand, if particles are transferred to the two neighbors ahead in even portions, branching of the avalanche occurs without gain or loss of particles.

In the general case we have determined exactly a number of important maximum possible values of: the current at any given odd, (30), and even, (31), moment of time; the occupation number ('height') of each site at any moment of time, see Proposition II. Our results for the maximum current reveal a quadratic increase with time τ, with leading asymptotic behavior $I_{max}(\tau) \propto \tau^2/2$. The leading asymptotic form of the maximum height is quadratic in time $I_{max}(\tau) \propto \tau^2/4$ at a finite distance from the cental site(s), while at a finite distance $d \geq 1$ from a closed boundary it changes to the linear one $h_{max}(\tau) \propto d\tau$, for d even, and $h_{max}(\tau) \propto (d-1)\tau$ for d odd. The above asymptotic laws have easy heuristic explanation in terms of number of particles involved in the relevant domain of sites.

Note that the maxima for the local heights are unconstrained, hence, in general, they do not happen simultaneously in any particular avalanche. The maximum height (occupation number) at a site, at a given virtual moment of time τ, is attained in particular avalanches which deplete to the maximum possible extent the basin of attraction of that site and focus

the flux of particles onto it. As a result, in these special avalanches all the remaining sites do not receive any particles at the moment τ, hence, they are left stable.

The above extreme values have been established on fully occupied stable configurations in a region of the lattice, the probability of which in the stationary state vanishes with the time τ as $2^{-\tau^2/2}$ for the maximum current, or $2^{-\tau^2/4}$ for the maximum height. The statistical weight of such events depends crucially on the still unsolved problem for obtaining the probability of all different avalanches at which the given values are realized.

As far as the temporal dependence of the averaged in the stationary state of the system avalanche front width $w_{\mathrm{av}}(\tau)$ and mean site occupation number $h_{\mathrm{av}}(\tau)$ are concerned, the existing theories [10], [11], as well as the simple random walk picture [13] and computer simulations [8], [12], agree upon the scaling laws $w_{\mathrm{av}}(\tau) \propto \tau^{1/2}$ and $h_{\mathrm{av}}(\tau) \propto \tau^{1/4}$. Obviously, these predictions pertain only to the stage of growth of the avalanches.

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